Isolated eigenvalues of linear operator and perturbations

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Throughout this talk let $X$ be a Banach space and $B(X)$ be the set of bounded linear operators acting on $X$.

For $T \in B(X)$ the spectrum of $T$ is defined by

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible} \}$$

and the resolvent set

$$\rho(T) = \mathbb{C} \setminus \sigma(T).$$

The complex number $\lambda$ is called an eigenvalue of $T$ if exists a non-zero vector $x \in X$ such that $T x = \lambda x$ (or equivalent $(T - \lambda I)x = 0$). The set of all eigenvalues of $T$ we denote $\sigma_p(T)$. 
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The complex number $\lambda$ is called an eigenvalue of $T$ if exists a non-zero vector $x \in X$ such that $Tx = \lambda x$ (or equivalent $(T - \lambda I)x = 0$). The set of all eigenvalues of $T$ we denote $\sigma_p(T)$. 
\[ \Lambda \subset \sigma(T) \] is called spectral set for \( T \) if both \( \Lambda \) and \( \sigma(T) \setminus \Lambda \) are closed in relative topology of \( \sigma(T) \).

For a spectral set \( \Lambda \) of \( T \) with \( C(T, \Lambda) \) we denote the set of all Cauchy contour \( C \) which separate \( \Lambda \) from \( \sigma(T) \setminus \Lambda \).

For \( z \in \rho(T) \),

\[ R(T, z) = (T - zI)^{-1} \]

is called the resolvent operator of \( T \) at \( z \).
The set $\Lambda \subset \sigma(T)$ is called spectral set for $T$ if both $\Lambda$ and $\sigma(T) \setminus \Lambda$ are closed in relative topology of $\sigma(T)$.

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    is called the resolvent operator of \( T \) at \( z \).
\end{itemize}
For a spectral set $\Lambda$ for $T \in B(X)$ and $C \in C(T, \Lambda)$, define

$$P(T, \Lambda) = -\frac{1}{2\pi i} \int_C R(T, z)dz$$

a bounded projection of $T$ and $\Lambda$.

For $\lambda \in \sigma(T)$ we say that is Riesz point of $T$ if $\lambda$ is an isolated eigenvalues of $T$ of finite algebraic multiplicity (or $\dim P(T, \lambda)(X) < \infty$).

For an isolated eigenvalue $\lambda$ of $T$ we say that has finite geometric multiplicity if $\dim N(T - \lambda I) < \infty$.

Let $\pi_0(T)$ denote the set of Riesz points of $T$ and let $\pi_{00}(T)$ denote the set of eigenvalues of $T$ of finite geometric multiplicity.
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Let $\pi_0(T)$ denote the set of Riesz points of $T$ and let $\pi_{00}(T)$ denote the set of eigenvalues of $T$ of finite geometric multiplicity.
It is known that $\pi_0(T) \subset \pi_{00}(T)$.

An eigenvalue $\lambda \in \pi_0(T)$ is called simple eigenvalues if $\dim P(T, \lambda) = 1$, or equivalent $X = \mathcal{N}(T - \lambda I) \oplus \mathcal{R}(T - \lambda I)$. 
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Motivation: Approximation of simple eigenvalues for bounded operators

- **Remark.** If \( \lambda \in \pi_0(T) \), then there exists a sequence \( \lambda_n \in \pi_0(T_n) \) such that \( \lambda_n \to \lambda \) and \( \dim P(T, \lambda) = \dim P(T_n, \lambda_n) \).

Refinement schemes for a simple eigenvalue

- **Theorem.** Let \( \lambda \) be a simple eigenvalue of \( T \) and \( \phi \) be a corresponding eigenvector. Assume that \( T_n \to T \).

Then for each large enough \( n \), \( T_n \) has a unique simple eigenvalue \( \lambda_n \) such that \( \lambda_n \to \lambda \).
**Remark.** If $\lambda \in \pi_0(T)$, then there exists a sequence $\lambda_n \in \pi_0(T_n)$ such that $\lambda_n \to \lambda$ and $\dim P(T, \lambda) = \dim P(T_n, \lambda_n)$.

**Refinement schemes for a simple eigenvalue**

**Theorem.** Let $\lambda$ be a simple eigenvalue of $T$ and $\phi$ be a corresponding eigenvector. Assume that $T_n \longrightarrow T$.

Then for each large enough $n$, $T_n$ has a unique simple eigenvalue $\lambda_n$ such that $\lambda_n \to \lambda$. 
Let \( \phi_n \) be an eigenvector of \( T_n \) corresponding to \( \lambda_n \) and \( \phi_n^* \) be the eigenvector of \( T_n^* \) corresponding to its eigenvalue \( \lambda_n^* \) such that \( \langle \phi_n, \phi_n^* \rangle = 1 \). Then \( \langle \phi, \phi_n^* \rangle \neq 0 \) for all large \( n \). If we let

\[
\phi(n) = \frac{\phi}{\langle \phi, \phi_n^* \rangle}
\]

then for all large \( n \), we have

\[
\max \left\{ |\lambda_n - \lambda|, \frac{\|\phi_n - \phi(n)\|}{\|\phi_n\|} \right\} \leq c\|T_n - T\|
\]

and if \( \lambda \neq 0 \), then

\[
\max \left\{ |\lambda_n - \lambda|, \frac{\|\phi_n - \phi(n)\|}{\|\phi_n\|} \right\} \leq c\|(T_n - T)T\|
\]

where \( c \) is a constant, independent of \( n \).
Finite rank approximations

Let $X$ be a complex Banach space and $T$ a bounded linear operator. With some extra conditions, for example if $T$ is a compact operator, or $X$ has Schauder basis, we can find $\{T_n\}$ a sequence of finite rank operators such that $T_n$ converge in norm to $T$. Since rank of operators $T_n$ are finite, then the spectral computations for $T_n$ can be reduced to solving a matrix eigenvalue problem in a canonical way. For this reason we will present various situations when we can apply this technics.

Examples

Approximation based on projections

Let \((\pi_n)\) be a sequence of bounded linear projection defined on a Banach space \(X\). Define:

\[ T_n^P = \pi_n T, \quad T_n^S = T \pi_n \text{ and } T_n^G = \pi_n T \pi_n. \]

The bounded operators \(T_n^P\), \(T_n^P\) and \(T_n^P\) are known as the projection approximation of \(T\), Sloan approximation of \(T\) and Galerkin approximation of \(T\), respectively.

**Theorem.** Let \(T \in B(X)\) and \(\pi_n(x) \to x(= I(x))\). Then

- If \(T\) is compact operator, then \(T_n^P \to T\);
- If \(T\) is compact operator and \(\pi_n^*(\cdot) \to I^*(\cdot)\), then \(T_n^S \to T\) and \(T_n^G \to T\).
Truncation of a Schauder expansion

Assume that $X$ has a Schauder basis $(e_i)$. For each positive integer $n$ define

$$\pi_n(x) = \sum_{j=1}^{n} c_j(x)e_j, \quad x \in X.$$ 

Then for $T \in B(X)$ such that $Te_j = \sum_{j=1}^{\infty} t_{i,j}e_i, \quad j = 1, 2, \ldots$ we have

$$T^P_n e_j = \sum_{j=1}^{n} t_{i,j}e_i, \quad j = 1, 2, \ldots$$

$$T^S_n e_j = \begin{cases} \sum_{j=1}^{\infty} t_{i,j}e_i, & j = 1, 2, \ldots, n \\ 0, & j > n. \end{cases}$$

$$T^G_n e_j = \begin{cases} \sum_{j=1}^{n} t_{i,j}e_i, & j = 1, 2, \ldots, n \\ 0, & j > n. \end{cases}$$
$K(X)$ denotes the ideal of all compact operators

- $\alpha(T) = \dim N(T); \quad \beta(T) = \dim(X/R(T))$

- $\phi_+(X) = \{T \in B(X) : R(T) \text{ is closed and } \alpha(T) < \infty\}$
  $\phi_-(X) = \{T \in B(X) : \beta(T) < \infty\}$

An operator $T \in B(H)$ is called semi-Fredholm if $T \in \phi_+(X) \cup \phi_-(X)$

An operator $T \in B(H)$ is called Fredholm if $T \in \phi_+(X) \cap \phi_-(X)$

The index of $T \in \phi_+(X) \cup \phi_-(X)$ is given by $\text{ind}(T) = \alpha(T) - \beta(T)$. 

 Localization of eigenvalues of linear operators

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Weyl’s theorem

- \( \phi_0(X) = \{ T \in B(X) : T \in \phi(X) \text{ and } \text{ind}(T) = 0 \} \) – Weyl operators

- Weyl spectrum by \( \sigma_w(A) = \{ \lambda \in \mathbb{C} : T - \lambda \notin \phi_0(X) \} \)

- \( \sigma_w(T) = \cap \{ \sigma(T + K) : K \in K(X) \} \)


- \( \lambda \) belongs to the spectra of all compact perturbations \( T + K \) of a single hermitian operator \( T \) if and only if \( \lambda \) is not an isolated eigenvalue of finite multiplicity

- We say that \( T \) obeys Weyl’s theorem if \( \sigma_w(T) = \sigma(T) \setminus \pi_{00}(T) \)

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Let $H$ be a Hilbert space. An operator $T \in B(H)$ is

- normal if $TT^* = T^*T$, or $\|T^*x\| = \|Tx\|$, for each $x \in H$,

- $T \in B(H)$ is hyponormal if $TT^* \leq T^*T$, or $\|T^*x\| \leq \|Tx\|$, for each $x \in H$,

- $T \in B(H)$ is $p$-hyponormal if $(T^*T)^p - TT^*)^p \geq 0$ holds,

- $T$ is quasi-hyponormal if $\|T^*Tx\| \leq \|T^2x\|$, for each $x \in H$.

**Theorem.** Let Hilbert space operators $T$ or $T^*$ are in one of the classes above. Then Weyl’s theorem holds for $T$ ($\pi_{00}(T) = \pi_0(T)$).
Some classes of operators

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Weyl's theorem for $T \rightsquigarrow$ Weyl's theorem for $T + K$
Weyl's theorem for $T \overset{?}{\rightarrow} T + K$

Theorem  Weyl's theorem is transmitted from $T \in B(X)$ to $T + N$, when $N$ is nilpotent operator commuting with $T$. 
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**Example** Let $X = \ell_2(\mathbb{N})$ and $T$ and $N$ be defined by:

\[
T(x_1, x_2, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots)
\]

\[
N(x_1, x_2, \ldots) = (0, -\frac{x_1}{2}, 0, \ldots).
\]

Then $T$ obeys Weyl's theorem, but $T + N$ not ($T$ and $N$ are not commuting).
Note In the previous example $N$ is also a finite rank operator not commuting with $T$. 
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**Example** In general, Weyl’s theorem is also not transmitted under commuting finite rank perturbation.

Let $X = \ell_2(\mathbb{N})$ and $S \in B(X)$ be an injective quasi-nilpotent operator and $U$ be defined:

$$U(x_1, x_2, \ldots) = (-x_1, 0, \ldots).$$

Define on $Y = X \oplus X$ the operators $T$ and $K$ by

$$T = \oplus S \quad \text{and} \quad K = U \oplus O.$$  

$K$ is a finite rank operator commuting with $T$, $T$ obeys Weyl’s theorem but $T + K$ not.

**Theorem** Suppose that $T \in B(H)$ is paranormal, $K$ algebraic and $TK = KT$. Then Weyl’s theorem is transmitted on $T + K$. 

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$T$ is paranormal if $\|Tx\|^2 \leq \|T^2x\| \cdot \|x\|$.

$K$ is algebraic if exists a polynomial $p$ such that $p(K) = 0$. 

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**Theorem** Suppose that $T \in B(H)$ satisfies Weyl’s theorem. If $\sigma(T)$ has no holes and has at most finitely many isolated points then Weyl’s theorem holds for $T + K$ for every compact operator $K$. 
Thank you.